"Matrix Wiener-Hopf Factorisation".

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§1 Introduction

The problem of the Wiener-Hopf factorisation of an arbitrary matrix still remains an open problem and one whose general solution would enable exact solutions to be found to many hitherto insoluble problems in acoustics and electromagnetic theory. Some progress has been made in recent years for some special cases and, in particular, Rawlins (1) and Hurd (2) have developed techniques which effectively lead to the Wiener-Hopf factorisation of particular matrices. The problems solved by these authors can also be solved, without recourse to Wiener-Hopf theory, by using the method of integral representations described in (3,4). The general approach developed by Hurd has however been shown, by Hurd and Prezdiecki (5), to be applicable to problems which do not appear to be soluble by the methods of (3,4).

Rawlins (1) and Hurd (2) reduce the problems under consideration to the solution of simultaneous Wiener-Hopf equations and, by making plausible assumptions, obtain functional equations for the components of the unknown vectors. The resulting functional equations can be identified with particular Riemann-Hilbert problems and a formal solution obtained. This formal approach is adopted in (2) but in (1) the functional equations, which are of a comparatively simple nature, are solved by an 'ad-hoc' method. The solutions obtained by Rawlins and Hurd can be shown to satisfy all the requisite conditions but, as has been pointed out by Chakrabarti (6), the approach suffers from the logical deficiency that equations valid in a strip in the complex plane are assumed, without apparent justification, to be valid in a half plane. This deficiency has been remedied in a recent paper (7) where the approach described in (1) is used to obtain functional equations for the elements of the factor matrices sought. The method adopted in (7) has also the advantage of demonstrating explicitly the required Wiener-Hopf factorisation, the problem solved therein can also be solved by using the method described in (3,4).

An alternative approach to the Wiener-Hopf factorisation of matrices has been recently devised by Daniele (8), who gives a method of factorising a class of matrices which includes those treated in (1), (2), (5) and (7) as special cases. The method used by Daniele is an algebraic one and its basis seems to be entirely different to that adopted in (1,2,5,7).

In this note we attempt to further extend the class of matrices for which an explicit Wiener-Hopf factorisation can be obtained and show that the technique described in (7) can be applied to a class of matrices, of which those in (1,2,7) are special cases, which does not appear to be equivalent to that class for which Daniele's approach is applicable. It is shown that, for the class of matrices considered, the problem of factorisation can be transformed to one of solving two independent Riemann-Hilbert problems on a half-line.

Factorisation procedure

The matrix A, which is to be factorised, is defined by

$$A = \begin{pmatrix} F(K) & G(K)F(K) \\ H(K) & -G(K)H(K) \end{pmatrix}, \qquad (1)$$

where F, G and H are analytic functions (except possibly at $K = 0$) of the variable K defined by $K = (k^2-\alpha^2)^{\frac{1}{2}}$, where α is a complex variable and k a constant with positive real and imaginary parts. The branch of the square root is chosen which has positive real part with the branch cuts being along the half-lines $\alpha = k+\delta$, $\alpha = -k-\delta\,(\delta \geq 0)$. The elements of A are therefore analytic functions of α within the strip $-k_i < \mathrm{Im}(\alpha) < k_i$, where k_i denotes the imaginary part of k, and the Wiener-Hopf matrix factorisation problem is the determination of matrices $U(\alpha)$ and $L(\alpha)$, whose elements are analytic for $Im(\alpha) > -k_i$ and $Im(\alpha) < k_i$ respectively, such that

$$A(\alpha) = U(\alpha)\ L^{-1}(\alpha). \tag{2}$$

We shall assume that F, G and H do not have any zeros for $R_e K > 0$ and that G has to satisfy

$$G(K) = -G(-K), \tag{3}$$

thus, unless G is unbounded at $K = 0$, $G(0) = 0$. The matrices studied in (2,3,4,7) are particular cases of the general type of matrix defined in equation (1) and for the particular example studied in detail in (7) we have that

$$F = K + k\beta,\ H = K,\ g = -i/K,$$

where β is a positive constant.

In order to effect the factorisation it will be assumed that U is analytic except along the branch cut through $\alpha = -k$ whilst L is analytic except along the branch cut through $\alpha = k$. Evaluation of equation (2) on both sides of the cut C through $\alpha = -k$ gives, on using the suffices $\pm$ to denote values evaluated on the upper and lower sides of C,

$$A_+(\alpha) = U_+(\alpha)\ L^{-1}(\alpha), \tag{4}$$

$$A_-(\alpha) = U_-(\alpha)\ L^{-1}(\alpha), \tag{5}$$

(L is analytic except across the branch cut through $\alpha = k$ and therefore takes the same values on both sides of C). Eliminating $L(\alpha)$ between equations (4) and (5) gives

$$U_+(\alpha) = A_+(\alpha)\quad A_-^{-1}(\alpha)\quad U_-(\alpha), \text{ on } C. \tag{6}$$

Equation (6) is a matrix Riemann-Hilbert problem and the columns of U are solutions of vector Riemann-Hilbert problems very similar to those encountered by Hurd (2). For k real the use of a suffix notation to describe values on appropriate sides of C can be dispensed with and we can, for example, write U_+ and U_- as $U(\xi e^{i\pi})$ and $U(\xi e^{-i\pi})$, respectively, where $k < \xi < \infty$. In this notation equation (6) becomes

$$U(\xi e^{i\pi}) = A(\xi e^{i\pi})\ A^{-1}(\xi e^{-i\pi})\ U(\xi e^{-\ \pi}), \tag{7}$$

which is a set of coupled q - difference equations and in (7) the factorisation is reduced to a set of difference equations similar to those defined by equation (7). It is to a certain extent a matter of taste as to the notation adopted (the results obtained for real k can be genralised to complex values by analytic continuation) and we adopt the notation of equation (6) as it enables results for the solution of Riemann-Hilbert problems to be quoted directly. (The difference equations associated with equation (7) can be solved by using the Plemelj formula in exactly the same way as this formula is used to solve Riemann-Hilbert problems). The values of K are different on the upper and lower surfaces of C and we let K denote the value on the upper side of C so that -K denotes its value on the lower side. Equation (7) is then seen to simplify to

$$U_+ = \begin{pmatrix} 0 & \dfrac{-F(K)}{H(-K)} \\ \dfrac{-H(K)}{F(-K)} & 0 \end{pmatrix} U_- . \tag{8}$$

Equation (8), when written in component form, is equivalent to the four scalar equations

$$u_{11}^+ = \frac{-F(K)}{H(-K)}\, u_{21}^-, \quad (9) \qquad u_{21}^+ = \frac{-H(K)}{F(-K)}\, u_{11}^-, \quad (10)$$

$$u_{12}^+ = \frac{-F(K)}{H(-K)}\, u_{22}^-, \quad (11) \qquad u_{22}^+ = \frac{-H(K)}{F(-K)}\, u_{12}^-, \quad (12)$$

where $u_{ij}^{\pm}$ are the elements of $U_{\pm}$ respectively. Equations (9) to (12) can clearly be solved if the coupled system

$$V_1^+ = \frac{-F(K)}{H(-K)}\, V_2^-, \quad (13) \qquad V_2^+ = \frac{-H(K)}{F(-K)}\, V_1^-, \quad (14)$$

can be solved. The system of equations (13) and (14) can be de-coupled to give

$$W_1^+ = \frac{F(K)H(K)}{F(-K)H(-K)} W_1^-, \tag{15}$$

$$W_2^- W_2^+ = \frac{F(K)F(-K)}{H(K)H(-K)}, \tag{16}$$

where $W_1 = V_1V_2$, $W_2 = V_1/V_2$.

Equation (15) is a homogeneous Riemann-Hilbert problem for W_1 whilst equation (16) becomes, on taking logarithms and defining

$$W_3 = (k+\alpha)^{\frac{1}{2}} \log W_2,$$

$$W_3^+ - W_3^- = (k+\alpha)^{\frac{1}{2}} \log \frac{F(K)F(-K)}{H(K)H(-K)}, \tag{17}$$

which is an inhomogeneous Riemann-Hilbert problem for W_3. Thus the original factorisation problem has been reduced to the solution of the Riemann-Hilbert problems defined by equations (15) and (17).

For given F, H, particular solutions of equations (15) and (17) can be obtained by standard methods and particular solutions of equations (13) and (14) can therefore be found. More general solutions of the latter equations are obtained by multiplying the particular solutions by solutions of

$$V_1^+ = V_2^-, \quad (18) \qquad V_2^+ = V_1^-, \quad (19)$$

and sufficiently general solutions of equations (18) and (19) are given by

$$V_1 = (k+\alpha)^{\frac{1}{2}n}, \quad (20) \qquad V_2 = (-1)^n(k+\alpha)^{\frac{1}{2}n}, \quad (21)$$

with n being any integer. The arbitrariness in the determination of the u_{ij} is a direct consequence of the fact that both U and L can be post-multiplied by a matrix whose coefficients are analytic functions of α. In any specific problem the arbitrariness can be reduced by placing restrictions on the behaviour of the coefficients of U and L as $|\alpha| \to \infty$.

The procedure described above effectively constructs a matrix U whose coefficents are analytic in the appropriate half plane and a matrix

L^{-1} whose coefficients are single-valued in the relevant half-plane. This constructional approach does not however ensure that the matrix L^{-1} is non-singular and that the elements of L are non-singular for $\mathrm{Re}\alpha < k$. This latter requirement implies that

$$\frac{u_{11}}{F} + \frac{u_{21}}{H}, \quad \frac{u_{12}}{F} + \frac{u_{22}}{H}, \quad \frac{u_{11}}{FG} - \frac{u_{21}}{GH}, \quad \frac{u_{12}}{FG} - \frac{u_{22}}{GH}$$

are analytic, our assumptions about F, G, H mean that these can only be zero for K = 0 and hence the values of n chosen in equations (20) and (21) must be such that the above expressions do not have single or multiple poles at $\alpha = -k$. (The construction has ensured that they will be single valued in a neighbourhood of $\alpha = -k$). The calculations can be a trifle tedious and are carried out in details for one particular case in (7).

The general steps necessary in the general calculation can however be illustrated without undue involvement in algebraic complexity by considering the special case when A is defined by

$$A = \begin{pmatrix} 1 & -K \\ K & K^2 \end{pmatrix} . \tag{22}$$

The matrix defined by equation (22) is that encountered in the direct Wiener-Hopf solution of the physical problem solved in (1). This latter problem corresponds to the case $\beta = \infty$ in the nomenclature of (7); the limiting process is non-uniform and the case $\beta = \infty$ has to be examined separately. Equations (15) and (16) become

$$W_1^+ = -W_1^-$$

$$W_2^- W_2^+ = \frac{-1}{K^2}$$

with particular solutions

$$W_1 = (k+\alpha)^{-\frac{1}{2}}, \; W_2 = (k+\alpha)^{-\frac{1}{2}} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{-1}.$$

Therefore more general solutions of equations (9) to (12) are given by

$$u_{11} = (k+\alpha)^{\frac{1}{2}n-\frac{1}{2}} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{-\frac{1}{2}}$$

$$u_{21} = (-1)^n(k+\alpha)^{\frac{1}{2}n} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{\frac{1}{2}},$$

$$u_{12} = (k+\alpha)^{\frac{1}{2}m-\frac{1}{2}} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{-\frac{1}{2}},$$

$$u_{22} = (-1)^m(k+\alpha)^{\frac{1}{2}m} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{\frac{1}{2}}.$$

(Two separate integers m and n have been introduced as the pairs of equations (9) and (10), (11) and (12) are independent and the values of the integer in equations (20) and (21) which is appropriate for solving the pair (9) and (10) need not be the same as that suitable for solving (11) and (12).)

The matrix A is non-singular and hence U being non-singular will ensure that L^{-1} is also non-singular. Direct evaluation of the determinant of U then gives m-n to be an odd integer. Examination of the behaviour of the ℓ_{ij} near $\alpha = -k$ gives that there will be no poles at $\alpha = -k$ provided that both $m \geq 1$ and $n \geq 1$. As $|\alpha| \to \infty$ we have

$$u_{11} = O(\alpha^{\frac{1}{2}n-\frac{3}{4}}), \qquad u_{21} = O(\alpha^{\frac{1}{2}n+\frac{1}{4}}),$$

$$u_{12} = O(\alpha^{\frac{1}{2}m-\frac{3}{4}}), \qquad u_{22} = O(\alpha^{\frac{1}{2}m+\frac{1}{4}}),$$

$$\ell_{11} = O(\alpha^{\frac{1}{2}n-\frac{3}{4}}), \qquad \ell_{12} = O(\alpha^{\frac{1}{2}m-\frac{3}{4}}),$$

$$\ell_{21} = O(\alpha^{\frac{1}{2}n-\frac{7}{4}}), \qquad \ell_{22} = O(\alpha^{\frac{1}{2}m-\frac{7}{4}}).$$

Thus, if we restrict u_{11} to be $o(1)$ and u_{12} to be $o(\alpha^{\frac{1}{2}})$ as $|\alpha| \to \infty$, n = 1 and m = 2.

References

(1) Rawlins, A.D. Proc. Roy. Soc. A 346, (1975), p 469

(2) Hurd, R.A. Canad. J. Phys. 54, (1976), p 775

(3) Williams W.E. Proc. Roy. Soc. A 252, (1959) p 376

(4) Maliushinetz, G.D. Soviet Physics (Acoustics) 1, (1955), p 152

(5) Hurd, R.A. and Presdiecki, S. Canad. J. Phys. 55, (1977), p 305

(6) Chakrabarti, A. Zeit. fur. Math. und Mech. 59, (1979), p 241

(7) Rawlins, A.D. - to be published

(8) Daniele, V. I.E.E. Trans. Antenna. and Propn. AP-26, (1978), p 614

Therefore more general solutions of equations (9) to (12) are given by

$$u_{11} = (k+\alpha)^{\frac{1}{2}n-\frac{1}{2}} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{-\frac{1}{2}}$$

$$u_{21} = (-1)^{n}(k+\alpha)^{\frac{1}{2}n} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{\frac{1}{2}} ,$$

$$u_{12} = (k+\alpha)^{\frac{1}{2}m-\frac{1}{2}} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{-\frac{1}{2}} ,$$

$$u_{22} = (-1)^{m}(k+\alpha)^{\frac{1}{2}m} [(2k)^{\frac{1}{2}} + (k+\alpha)^{\frac{1}{2}}]^{\frac{1}{2}} .$$

(Two separate integers m and n have been introduced as the pairs of equations (9) and (10), (11) and (12) are independent and the values of the integer in equations (2) and (21) which is appropriate for solving the pair (9) and (10) need not be the same as that suitable for solving (11) and (12).)

The matrix A is non-singular and hence U being non-singular will ensure that L^{-1} is also non-singular. Direct evaluation of the determinant of U then gives m-n to be an odd integer. Direct evaluation of the determinant of L gives that $m+n = 3$. Examination of the behaviour of the ℓ_{ij} near $\alpha = -k$ gives that there will be no poles at $\alpha = -k$ provided that both $m \geq 1$ and $n \geq 1$. Thus the only possible values of m and n are $m = 1$, $n = 2$ or $m = 2$, $n = 1$. Either of these sets of values can be used because the matrices will be unique to within an arbitrary constant matrix, and the product UL^{-1} will equal A.